S.-T. Yau College Student Mathematics Contests 2018

Analysis and Differential Equations

Team

Please solve the following problems.

1. Isoperimetric inequality. and Steiner symmetrization.

It is well known that in $\mathbb{R}^2$, any region Ω with continuous piecewise C^1 boundary $\partial\Omega$ satisfies that

$$4\pi|\Omega| \leq \text{Length}(\partial\Omega)^2.$$

When the equality holds for a region Ω with continuous piecewise C^1 boundary, it is then called an isoperimetric set. For example, all disks are isoperimetric sets.

1) Prove that any isoperimetric set is convex;

2) Given a line V in $\mathbb{R}^2$ passing through the origin, the Steiner symmetrization with respect to V is the operation which associates with each bounded convex subset C of $\mathbb{R}^2$ the subset $S(C)$ of $\mathbb{R}^2$ such that, for every line L (not necessarily passing through the origin) perpendicular to V,

either $L \cap C = \emptyset$ and $L \cap S(C) = \emptyset$,

or $L \cap C \neq \emptyset$ and $L \cap S(C)$ is a closed segment with center in V, and Length($L \cap S(C)$)=Length($L \cap C$). In other words, $L \cap S(C)$ is a translation of the segment $L \cap C$ along L, such that the segment $L \cap S(C)$ is symmetric with respect to V.

Prove that $|S(C)| = |C|$, and Length($\partial S(C)$) $\geq$Length(C), equality holds if and only if C is symmetric with respect to V.

3) Deduce that the only isoperimetric sets are the disks.

2. Define $\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}, s = \sigma + it$. Show that

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z - 1}dz$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx, (-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < Im\log(-z) < \pi.C$ is the contor consisting of upper real axis in the direction from $+\infty$ to 0, a small circle around 0 counter-clockwise, and lower real axis in the direction from 0 to $+\infty$.

Show that $\zeta(s)$ is a meromorphic function of the complex plane.

3. Consider the following linear parabolic equation

$$\begin{cases} \partial_t u(t;r,z) = (\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u(t;r,z), \text{ on } \mathbb{R}^+ \times \Omega \\ u(0;r,z) = u_0(r,z), u(t;0,z) = 0 \end{cases}$$

where $\Omega = \{(r,z); r > 0, z \in \mathbb{R}\}$. Given initial data u_0 satisfying

$$\int_\Omega |u_0| drdz < \infty.$$

Prove that $u(t;r,z)$ is given by

$$u(t;r,z) = \frac{1}{4\pi t}\int_\Omega |\frac{r'}{r}|^{\frac{1}{2}} H(\frac{t}{rr'})\exp(-\frac{(r-r')^2 + (z-z')^2}{4t})u_0(r',z')dr'dz'$$

where

$$H(y) = \frac{1}{\sqrt{\pi y}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\frac{\sin^2\phi}{y}}\cos(2\phi)d\phi$$

4. Let Ω be a bounded and connected smooth domain in $\mathbb{R}^2$, and $\tilde{N} = (\tilde{N}_1, \tilde{N}_2)$ be a smooth vector field defined on $\bar{\Omega}$ that equals to the outer unit normal N on the boundary $\partial\Omega$ satisfying $\sup_{x\in\bar{\Omega}} |\tilde{N}|(x) \leq 1$ and $\sup_{x\in\bar{\Omega}} |\partial_{x_j}\tilde{N}|(x) \leq K$ $(j = 1,2)$ for some constant K. Prove that there exists a constant C independent of K such that

$$\int_\Omega \sum_{i,j=1}^{2} |\partial_i v_j|^2 dx$$

$$\leq C\int_\Omega \sum_{1\leq i,k,l\leq 2} (\tilde{N}^k\tilde{N}^l\partial_i v_k\partial_i v_l + |curl v|^2 + |\mathrm{div} v|^2 + K|v|^2)(x)dx,$$

for any smooth vector field $v = (v_1, v_2)$ on $\bar{\Omega}$, where $|curl v|^2 = \sum_{i,j=1}^{2}(\partial_i v_j - \partial_j v_i)^2$, $|\mathrm{div} v|^2 = \sum_{i=1}^{2}(\partial_i v_i)^2$.